

10. cvičení - řešení

Poznámka k racionální funkci R u goniometrické substituce níže: často stačí jen určit podobu R a pak ověřovat rovnosti $R(-u, v) = -R(u, v)$ apod. Když platí tyhle rovnosti, tak platí i odpovídající rovnici, kdy za u volíme $\sin x$ a za v volíme $\cos x$. Nicméně se teoreticky může stát, že rovnost $R(u, v) = -R(-u, v)$ splněna není, ale $R(\sin x, \cos x) = -R(-\sin x, \cos x)$. Zároveň mi přijde více polopatické/pochopitelnější ověřovat přímo vztah s těmi goniometrickými funkcemi.

Příklad 1 (a)

$\frac{1}{\sin x}$ je definováno pro $x \neq k\pi, k \in \mathbb{Z}$. Má smysl tedy integrovat jen na intervalech $(k\pi, (k+1)\pi), k \in \mathbb{Z}$.

Jak přijít na vhodnou substituci: Integrujeme funkci $\frac{1}{\sin x}$. Chceme použít postup z teorie, tedy na tuto funkci chceme nahlížet jako na racionální (polynom lomeno polynom) funkci o proměnných $\sin x$ a $\cos x$. Tj. chceme najít racionální funkci R takovou, že $R(\sin x, \cos x) = \frac{1}{\sin x}$. Jaká racionální funkce to splňuje? Tato: $R(u, v) = \frac{1}{u}$. Nyní chci zjistit, který z případů v teorii je splněn - tj. podívám se postupně na $R(-\sin x, \cos x)$, $R(\sin x, -\cos x)$ a $R(-\sin x, -\cos x)$ a popřemýšlím nad tím, jestli je splněn nějaký příklad z rozboru v teorii.

Výraz	Čemu je roven	Čemu chci, aby se to rovnalo	Platí daný vztah?
$R(-\sin x, \cos x)$	$\frac{1}{-\sin x}$	$-\frac{1}{\sin x}$	ano

Poznámka k výpočtu níže: Došli jsme k tomu, že chceme použít substituci $y = \cos x$, $dy = -\sin x \, dx$. Potřebujeme v integrálu tedy najít $-\sin x \, dx$. Bohužel to tam nemáme, takže si to tam musíme vyrobit - zde tak, že výraz přenásobíme jedničkou $\frac{-\sin x}{-\sin x}$. Tím jsme vyrobili, co jsme potřebovali. Dole nám vzniklo $-\sin^2 x$ a to potřebujeme vyjádřit pomocí y . My ale umíme substituovat jen $\cos x$, takže potřebujeme to $-\sin^2 x$ nějak přetvořit na něco, kde se objevuje jen kosinus a čísla. Využijeme proto Pythagorovu větu $1 = \sin^2 x + \cos^2 x$.

$$\begin{aligned} \int \frac{1}{\sin x} \, dx &= |y = \cos x, \, dy = -\sin x \, dx| = \int \frac{-\sin x}{-\sin^2 x} \, dx = \int \frac{-\sin x}{-(1 - \cos^2 x)} \, dx = \\ &= \int \frac{1}{y^2 - 1} \, dy \end{aligned}$$

Parciální zlomky: $\frac{1}{y^2-1} = \frac{A}{y-1} + \frac{B}{y+1} \implies 1 = Ay + A + By - B \implies A + B = 0, A - B = 1$

$$\begin{aligned} \int \frac{1}{y^2-1} \, dy &\stackrel{\text{lin.}}{=} -\frac{1}{2} \int \frac{1}{y+1} \, dy + \frac{1}{2} \int \frac{1}{y-1} \, dy \stackrel{c}{=} -\frac{1}{2} \log |y+1| + \frac{1}{2} \log |y-1| \\ &\implies \int \frac{1}{\sin x} \, dx \stackrel{c}{=} \frac{1}{2} \log |\cos x - 1| - \frac{1}{2} \log |\cos x + 1| \end{aligned}$$

Příklad 1 (b)

Zřejmě je vnitřek integrálu definován pro všechna $x \in \mathbb{R}$.

Výběr substituce:

Chci najít racionální fci R t.ž. $R(\sin x, \cos x) = \frac{1}{\sin^4 x + \cos^4 x}$. To je fce $R(u, v) = \frac{1}{u^4 + v^4}$.

Výraz	Čemu je roven	Čemu chci, aby se to rovnalo	Platí daný vztah?
$R(-\sin x, \cos x)$	$\frac{1}{(-\sin x)^4 + \cos^4 x}$	$-\frac{1}{\sin^4 x + \cos^4 x}$	ne
$R(\sin x, -\cos x)$	$\frac{1}{\sin^4 x + (-\cos x)^4}$	$-\frac{1}{\sin^4 x + \cos^4 x}$	ne
$R(-\sin x, -\cos x)$	$\frac{1}{(-\sin x)^4 + (-\cos x)^4}$	$\frac{1}{\sin^4 x + \cos^4 x}$	ano

Ověření vzorců pro substituci uvedeného v teorii:

$$\begin{aligned}\tan^2 x &= \frac{\sin^2 x}{\cos^2 x} \\ \tan^2 x + 1 &= \frac{1}{\cos^2 x} \\ \cos^2 x &= \frac{1}{\tan^2 x + 1} \\ \sin^2 x &= 1 - \cos^2 x = 1 - \frac{1}{\tan^2 x + 1} = \frac{\tan^2 x + 1 - 1}{\tan^2 x + 1} = \frac{\tan^2 x}{\tan^2 x + 1}\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\sin^4 x + \cos^4 x} dx &= \left| y = \tan x, dy = \frac{1}{\cos^2 x} dx \right| = \int \frac{\cos^2 x}{\sin^4 x + \cos^4 x} \cdot \frac{1}{\cos^2 x} dx = \\ &= \int \frac{\frac{1}{y^2+1}}{\left(\frac{y^2}{y^2+1}\right)^2 + \left(\frac{1}{y^2+1}\right)^2} dy = \int \frac{1}{y^2+1} \cdot \frac{(y^2+1)^2}{y^4+1} dy = \int \frac{y^2+1}{y^4+1} dy\end{aligned}$$

Vyýpočet:

Rozklad $y^4 + 1$ pomocí finty z reciprokových rovnic.

Vizte: https://www2.karlin.mff.cuni.cz/~portal/komplexni_cisla.dp/?page=rovnice-reciproke

$$\begin{aligned}y^4 + 1 &= y^2 \left(y^2 + \frac{1}{y^2} \right) \\ z = y + \frac{1}{y} &\implies z^2 = y^2 + 2 + \frac{1}{y^2} \\ \implies y^2 + \frac{1}{y^2} &= z^2 - 2 = (z - \sqrt{2})(z + \sqrt{2}) \\ \implies y^4 + 1 &= y^2 \left(y^2 + \frac{1}{y^2} \right) = y^2 (z - \sqrt{2})(z + \sqrt{2}) = y \left(y + \frac{1}{y} - \sqrt{2} \right) y \left(y + \frac{1}{y} + \sqrt{2} \right) = \\ &= (y^2 - y\sqrt{2} + 1)(y^2 + y\sqrt{2} + 1)\end{aligned}$$

Trojčleny $y^2 \pm y\sqrt{2} + 1$ již nelze rozložit (záporný diskriminant).

Parciální zlomky:

$$\begin{aligned}\frac{y^2 + 1}{(y^2 - y\sqrt{2} + 1)(y^2 + y\sqrt{2} + 1)} &= \frac{Ay + B}{(y^2 + y\sqrt{2} + 1)} + \frac{Cy + D}{(y^2 - y\sqrt{2} + 1)} \\ y^2 + 1 &= (Ay + B)(y^2 - y\sqrt{2} + 1) + (Cy + D)(y^2 + y\sqrt{2} + 1) \\ y^2 + 1 &= Ay^3 - Ay^2\sqrt{2} + Ay + By^2 - By\sqrt{2} + B + Cy^3 + Cy^2\sqrt{2} + Cy + Dy^2 + Dy\sqrt{2} + D \\ y^2 + 1 &= y^3(A + C) + y^2(-A\sqrt{2} + B + C\sqrt{2} + D) + y(A - B\sqrt{2} + C + D\sqrt{2}) + B + D\end{aligned}$$

Porovnáním koeficientů dostáváme:

$$\begin{aligned}
 y^3 &\implies 0 = A + C \implies -A = C \xrightarrow{2} A = 0 \\
 y^2 &\implies 1 = -A\sqrt{2} + B + C\sqrt{2} + D \xrightarrow{1} 1 = 2C\sqrt{2} + B + D \xrightarrow{4} C = 0 \\
 y &\implies 0 = A - B\sqrt{2} + C + D\sqrt{2} \xrightarrow{1+2} 0 = \sqrt{2}(D - B) \implies B = D \\
 y^0 &\implies 1 = B + D \xrightarrow{4} B = D = \frac{1}{2}
 \end{aligned}$$

Máme tedy: $A = C = 0, B = D = \frac{1}{2}$.

$$\begin{aligned}
 \int \frac{y^2 + 1}{y^4 + 1} dy &\stackrel{\text{lin.}}{=} \frac{1}{2} \int \frac{1}{y^2 + y\sqrt{2} + 1} dy + \frac{1}{2} \int \frac{1}{y^2 - y\sqrt{2} + 1} dy = \\
 &= \frac{1}{2} \int \frac{1}{\left(y + \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} dy + \frac{1}{2} \int \frac{1}{\left(y - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} dy = \\
 &= \int \frac{1}{\left(\frac{y + \frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}}\right)^2 + 1} dy + \int \frac{1}{\left(\frac{y - \frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}}\right)^2 + 1} dy = \\
 &= \left| z = \sqrt{2} \left(y + \frac{\sqrt{2}}{2}\right), dz = \sqrt{2} dy, u = \sqrt{2} \left(y - \frac{\sqrt{2}}{2}\right), du = \sqrt{2} dy \right| = \\
 &= \frac{1}{\sqrt{2}} \int \frac{1}{z^2 + 1} dz + \frac{1}{\sqrt{2}} \int \frac{1}{u^2 + 1} du \stackrel{c}{=} \frac{1}{\sqrt{2}} \arctan z + \frac{1}{\sqrt{2}} \arctan u = \\
 &\stackrel{c}{=} \frac{1}{\sqrt{2}} \arctan \sqrt{2} \left(y + \frac{\sqrt{2}}{2}\right) + \frac{1}{\sqrt{2}} \arctan \sqrt{2} \left(y - \frac{\sqrt{2}}{2}\right) = \\
 &\stackrel{c}{=} \frac{1}{\sqrt{2}} \arctan \sqrt{2} \left(\tan x + \frac{\sqrt{2}}{2}\right) + \frac{1}{\sqrt{2}} \arctan \sqrt{2} \left(\tan x - \frac{\sqrt{2}}{2}\right) = \\
 &\stackrel{c}{=} \frac{1}{\sqrt{2}} \arctan \left(\sqrt{2} \tan x + 1\right) + \frac{1}{\sqrt{2}} \arctan \left(\sqrt{2} \tan x - 1\right)
 \end{aligned}$$

Příklad 1 (c) $\int \frac{\cos^3 x}{2 - \sin x} dx$

Podmínky: $\sin x \neq 2$ - platí vždy.

Jak zvolit substituci:

Chci $R(\sin x, \cos x) = \frac{\cos^3 x}{2 - \sin x}$. Tj. máme $R(u, v) = \frac{v^3}{1 - u}$.

Výraz	Čemu je roven	Čemu chci, aby se to rovnalo	Platí daný vztah?
$R(-\sin x, \cos x)$	$\frac{\cos^3 x}{1 - (-\sin x)}$	$-\frac{\cos^3 x}{1 - \sin x}$	ne
$R(\sin x, -\cos x)$	$\frac{(-\cos x)^3}{1 - \sin x}$	$-\frac{\cos^3 x}{1 - \sin x}$	ano

$$\begin{aligned}
 \int \frac{\cos^3 x}{2 - \sin x} dx &= |y = \sin x, dy = \cos x dx| = \int \frac{(1 - \sin^2 x) \cos x}{2 - \sin x} dx = \int \frac{1 - y^2}{2 - y} dy = \\
 &= \int \frac{y^2 - 1}{y - 2} dy = \int y + 2 + \frac{3}{y - 2} dy = |z = y - 2, dz = dy| \stackrel{\text{lin.}}{=} \frac{y^2}{2} + 2y + 3 \int \frac{1}{z} dz = \\
 &\stackrel{c}{=} \frac{y^2}{2} + 2y + 3 \log |z| \stackrel{c}{=} \frac{y^2}{2} + 2y + 3 \log |y - 2| \stackrel{c}{=} \frac{\sin^2 x}{2} + 2 \sin x + 3 \log |\sin x - 2|
 \end{aligned}$$

Příklad 1 (d) $\int \frac{1}{2-\cos x} dx$

Všimněme si, že $\frac{1}{2-\cos x}$ je definováno všude.

Jak přijít na substituci:

Chceme $R(\sin x, \cos x) = \frac{1}{2-\cos x}$. Tj. $R(u, v) = \frac{1}{2-v}$.

Výraz	Čemu je roven	Čemu chci, aby se to rovnalo	Platí daný vztah?
$R(-\sin x, \cos x)$	$\frac{1}{2-\cos x}$	$-\frac{1}{2-\cos x}$	ne
$R(\sin x, -\cos x)$	$\frac{1}{2-(-\cos x)}$	$-\frac{1}{2-\cos x}$	ne
$R(-\sin x, -\cos x)$	$\frac{1}{2-(-\cos x)}$	$\frac{1}{2-\cos x}$	ne

Nenastává ani jeden z prvních tří případů, musíme proto volit substituci $y = \tan \frac{x}{2}$.

Ověření vzorců pro substituci:

$$t = \tan \frac{x}{2}$$

$$\tan^2 \frac{x}{2} = \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} \implies 1 + \tan^2 \frac{x}{2} = 1 + \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} = \frac{1}{\cos^2 \frac{x}{2}} \quad (\text{Pythagorova věta})$$

$$\implies \cos^2 \frac{x}{2} = \frac{1}{1 + \tan^2 \frac{x}{2}}$$

$$dt = \frac{1}{\cos^2 \frac{x}{2}} \cdot \frac{1}{2} dx = \frac{1}{2} \left(1 + \tan^2 \frac{x}{2} \right) dx = \frac{1}{2} (1 + t^2) dx \implies \frac{2}{1 + t^2} dt = dx$$

$$\sin x = \sin \left(2 \frac{x}{2} \right) = 2 \sin \frac{x}{2} \cos \frac{x}{2} = 2 \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \cos^2 \frac{x}{2} = 2 \tan \frac{x}{2} \cdot \frac{1}{1 + \tan^2 \frac{x}{2}} = \frac{2t}{1 + t^2}$$

$$\begin{aligned} \cos x &= \cos \left(2 \frac{x}{2} \right) = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos^2 \frac{x}{2} - \left(1 - \cos^2 \frac{x}{2} \right) = \frac{1}{1 + t^2} - \left(1 - \frac{1}{1 + t^2} \right) = \\ &= \frac{1}{1 + t^2} - \frac{1 + t^2 - 1}{1 + t^2} = \frac{1 - t^2}{1 + t^2} \end{aligned}$$

Výpočet:

$$\begin{aligned} \int \frac{1}{2 - \cos x} dx &= \left| t = \tan \frac{x}{2}, \frac{2}{1 + t^2} dt = dx \right| = \int \frac{1}{2 - \frac{1-t^2}{1+t^2}} \frac{2}{1 + t^2} dt = \int \frac{1}{\frac{2+2t^2-1+t^2}{1+t^2}} \frac{2}{1 + t^2} dt = \\ &= \int \frac{1 + t^2}{1 + 3t^2} \frac{2}{1 + t^2} dt = \int \frac{2}{(\sqrt{3}t)^2 + 1} dt = \left| y = \sqrt{3}t, dy = \sqrt{3} dt \right| = \\ &\stackrel{\text{lin.}}{=} \frac{2}{\sqrt{3}} \int \frac{1}{y^2 + 1} dy \stackrel{c}{=} \frac{2}{\sqrt{3}} \arctan y \stackrel{c}{=} \frac{2}{\sqrt{3}} \arctan (\sqrt{3}t) \stackrel{c}{=} \frac{2}{\sqrt{3}} \arctan \left(\sqrt{3} \tan \frac{x}{2} \right) \end{aligned}$$

Příklad 1 (e) $\int \frac{1}{1+\sin^2 x} dx$

Výběr substituce:

Chci: $R(\sin x, \cos x) = \frac{1}{1+\sin^2 x}$. Tj. $R(u, v) = \frac{1}{1+u^2}$.

Výraz	Čemu je roven	Čemu chci, aby se to rovnalo	Platí daný vztah?
$R(-\sin x, \cos x)$	$\frac{1}{1+(\sin x)^2}$	$-\frac{1}{1+\sin^2 x}$	ne
$R(\sin x, -\cos x)$	$\frac{1}{1+\sin^2 x}$	$-\frac{1}{1+\sin^2 x}$	ne
$R(-\sin x, -\cos x)$	$\frac{1}{1+(-\sin x)^2}$	$\frac{1}{1+\sin^2 x}$	ano

Volme tedy substituci $t = \tan x$.

$$\begin{aligned}
\int \frac{1}{1+\sin^2 x} dx &= \left| t = \tan x, dx = \frac{1}{1+t^2} dt \right| = \int \frac{1}{1+\frac{t^2}{1+t^2}} \cdot \frac{1}{1+t^2} dt = \int \frac{t^2+1}{1+t^2+t^2} \frac{1}{1+t^2} dt = \\
&= \frac{1}{1+2t^2} dt = \int \frac{1}{1+(t\sqrt{2})^2} dt = \left| y = t\sqrt{2}, dy = \sqrt{2} dt \right| \stackrel{\text{lin.}}{=} \frac{1}{\sqrt{2}} \int \frac{1}{1+y^2} dy = \\
&\stackrel{c}{=} \frac{1}{\sqrt{2}} \arctan y \stackrel{c}{=} \frac{1}{\sqrt{2}} \arctan(\sqrt{2}t) \stackrel{c}{=} \frac{1}{\sqrt{2}} \arctan(\sqrt{2}\tan x)
\end{aligned}$$

Příklad 1 (f) $\int \frac{1}{\cos x \cdot \sin^3 x} dx$

Podmínky: $\cos x \neq 0 \wedge \sin x \neq 0 \implies x \neq k\frac{\pi}{2}, k \in \mathbb{Z}$.

Výběr substituce:

Chci: $R(\sin x, \cos x) = \frac{1}{\cos x \cdot \sin^3 x}$. Tj. $R(u, v) = \frac{1}{v \cdot u^3}$.

Výraz	Čemu je roven	Čemu chci, aby se to rovnalo	Platí daný vztah?
$R(-\sin x, \cos x)$	$\frac{1}{\cos x \cdot (-\sin x)^3}$	$-\frac{1}{\cos x \cdot \sin^3 x}$	ano

$$\begin{aligned}
\int \frac{1}{\cos x \cdot \sin^3 x} dx &= |t = \cos x, dt = -\sin x dx| = \int \frac{-\sin x}{-\cos x \cdot \sin^4 x} dx = \\
&= \int \frac{-\sin x}{-\cos x \cdot (1 - \cos^2 x)^2} dx = \int \frac{-1}{t \cdot (1 - t^2)^2} dt
\end{aligned}$$

Parciální zlomky:

$$\begin{aligned}
\frac{-1}{t(t^2-1)^2} &= \frac{-1}{t(t+1)^2(t-1)^2} = \frac{A}{t} + \frac{B}{t+1} + \frac{C}{(t+1)^2} + \frac{D}{t-1} + \frac{E}{(t-1)^2} \\
-1 &= A(t^2-1)^2 + Bt(t+1)(t-1)^2 + Ct(t-1)^2 + Dt(t-1)(t+1)^2 + Et(t+1)^2 \\
t=0 &\implies -1 = A(-1)^2 \implies A = -1 \\
t=1 &\implies -1 = E4 \implies E = -\frac{1}{4} \\
t=-1 &\implies -1 = C(-1)(-2)^2 \implies C = \frac{1}{4} \\
-1 &= -(t^2-1)^2 + Bt(t+1)(t-1)^2 + \frac{1}{4}t(t-1)^2 + Dt(t-1)(t+1)^2 - \frac{1}{4}t(t+1)^2 \\
-1 &= -t^4 + 2t^2 - 1 + Bt^4 + \left(B + \frac{1}{4}\right)t^3 - 2Bt^3 - 2\left(B + \frac{1}{4}\right)t^2 + Bt^2 + \left(B + \frac{1}{4}\right)t + Dt^4 + \\
&\quad + \left(-\frac{1}{4} - D\right)t^3 + 2Dt^3 + 2\left(-\frac{1}{4} - D\right)t^2 + Dt^2 + \left(-\frac{1}{4} - D\right)t \\
-1 &= t^4(-1 + B + D) + t^3\left(B + \frac{1}{4} - 2B - \frac{1}{4} - D + 2D\right) + \\
&\quad + t^2\left(2 - 2B - 2\frac{1}{4} + B - 2\frac{1}{4} - 2D + D\right) + t\left(B + \frac{1}{4} - \frac{1}{4} - D\right) - 1
\end{aligned}$$

Porovnáním koeficientů dostáváme soustavu:

$$\begin{aligned}
-1 + B + D &= 0 \implies B + D = 1 \xrightarrow{2} 2D = \frac{1}{2} \implies B = \frac{1}{2} = D \\
-B + D &= 0 \implies B = D \\
1 - B - D &= 0 \\
B - D &= 0
\end{aligned}$$

Řešení soustavy je: $A = -1, B = \frac{1}{2}, C = \frac{1}{4}, D = \frac{1}{2}, E = -\frac{1}{4}$

$$\begin{aligned}
\int \frac{-1}{t(1-t^2)^2} dt &= \int -\frac{1}{t} + \frac{1}{2} \frac{1}{t+1} + \frac{1}{4} \frac{1}{(t+1)^2} + \frac{1}{2} \frac{1}{t-1} - \frac{1}{4} \frac{1}{(t-1)^2} dt = \\
&= |\text{substituce}| \stackrel{c}{=} -\log|t| + \frac{1}{2} \log|t+1| - \frac{1}{4} \frac{1}{t+1} + \frac{1}{2} \log|t-1| + \frac{1}{4} \frac{1}{t-1} = \\
&\stackrel{c}{=} \frac{1}{4} \frac{t+1-(t-1)}{(t-1)(t+1)} + \frac{1}{2} \log|(t+1)(t-1)| - \log|t| = \\
&\stackrel{c}{=} \frac{1}{4} \frac{2}{t^2-1} + \frac{1}{2} (\log|t^2-1| - \log|t^2|) \stackrel{c}{=} \frac{1}{2(t^2-1)} + \frac{1}{2} \log \left| \frac{t^2-1}{t^2} \right| = \\
&\stackrel{c}{=} \frac{1}{2(\cos^2 x) - 1} + \log \sqrt{\frac{1 - \cos^2 x}{\cos^2 x}} \stackrel{c}{=} -\frac{1}{2 \sin^2 x} + \log \sqrt{\frac{\sin^2 x}{\cos^2 x}} = \\
&\stackrel{c}{=} -\frac{1}{2 \sin^2 x} + \log |\tan x|
\end{aligned}$$

Příklad 1 (g) $\int \frac{\sin x}{1+\sin x} dx$

Podmínky: $\sin x \neq -1 \implies x \neq \frac{3\pi}{2} + k2\pi, k \in \mathbb{Z}$.

Výběr substituce:

Chci: $R(\sin x, \cos x) = \frac{\sin x}{1+\sin x}$. Tj. $R(u, v) = \frac{u}{1+u}$.

Výraz	Čemu je roven	Čemu chci, aby se to rovnalo	Platí daný vztah?
$R(-\sin x, \cos x)$	$\frac{-\sin x}{1-\sin x}$	$-\frac{\sin x}{1+\sin x}$	ne
$R(\sin x, -\cos x)$	$\frac{\sin x}{1+\sin x}$	$-\frac{\sin x}{1+\sin x}$	ne
$R(-\sin x, -\cos x)$	$\frac{-\sin x}{1-\sin x}$	$\frac{\sin x}{1+\sin x}$	ne

$$\begin{aligned} \int \frac{\sin x}{1+\sin x} dx &= \left| t = \tan \frac{x}{2}, dx = \frac{2}{1+t^2} dt \right| = \int \frac{\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}} \frac{2}{1+t^2} dt = \int \frac{4t}{(1+t^2)^2} \cdot \frac{1+t^2}{1+t^2+2t} dt = \\ &= \int \frac{4t}{(t^2+1)(t+1)^2} dt \end{aligned}$$

Parciální zlomky:

$$\begin{aligned} \frac{4t}{(t^2+1)(t+1)^2} &= \frac{At+B}{t^2+1} + \frac{C}{t+1} + \frac{D}{(t+1)^2} \\ 4t &= (At+B)(t^2+1) + C(t+1)(t^2+1) + D(t^2+1) \\ t = -1 &\implies -4 = 2D \implies D = -2 \\ 4t &= At^3 + 2At^2 + At + Bt^2 + 2Bt + B + Ct^3 + Ct + Ct^2 + C - 2t^2 + -2 \end{aligned}$$

Porovnáním koeficientů dostáváme soustavu:

$$\begin{aligned} 0 &= A + C \implies C = -A \xrightarrow{4} C = 0 \\ 0 &= 2A + B + C - 2 \xrightarrow{1} 2 = A + B \implies B = 2 - A \xrightarrow{4} B = 2 \\ 4 &= A + 2B + C \xrightarrow{2} 4 = 2 + B + C \\ 0 &= B + C - 2 \xrightarrow{1} 2 = B - A \xrightarrow{2} 2 = 2 - 2A \implies A = 0 \end{aligned}$$

$$\begin{aligned} \int \frac{4t}{(t^2+1)(t+1)^2} dt &\stackrel{\text{lin.}}{=} \int \frac{2}{t^2+1} dt + \int \frac{-2}{(t+1)^2} dt = |y = t+1, dy = dt| = \\ &= 2 \arctan t - 2 \int \frac{1}{y^2} dy \stackrel{c}{=} 2 \arctan t + 2 \frac{1}{y} = \\ &\stackrel{c}{=} 2 \arctan t + 2 \frac{1}{t+1} \stackrel{c}{=} 2 \arctan \left(\tan \frac{x}{2} \right) + 2 \frac{1}{\tan \frac{x}{2} + 1} \stackrel{c}{=} 2 \frac{x}{2} + \frac{2}{\tan \frac{x}{2} + 1} = \\ &\stackrel{c}{=} x + \frac{2}{\tan \frac{x}{2} + 1} \end{aligned}$$

Příklad 1 (h) $\int \frac{2-\sin x}{2+\cos x} dx$
 $\frac{2-\sin x}{2+\cos x}$ je definováno všude.

Výběr substituce:

Chci: $R(\sin x, \cos x) = \frac{2-\sin x}{2+\cos x}$. Tj. $R(u, v) = \frac{2-u}{2+v}$.

Výraz	Čemu je roven	Čemu chci, aby se to rovnalo	Platí daný vztah?
$R(-\sin x, \cos x)$	$\frac{2-(-\sin x)}{2+\cos x}$	$-\frac{2-\sin x}{2+\cos x}$	ne
$R(\sin x, -\cos x)$	$\frac{2-\sin x}{2+(-\cos x)}$	$-\frac{2-\sin x}{2+\cos x}$	ne
$R(-\sin x, -\cos x)$	$\frac{2-(-\sin x)}{2+(-\cos x)}$	$\frac{2-\sin x}{2+\cos x}$	ne

$$\begin{aligned} \int \frac{2-\sin x}{2+\cos x} dx &= \left| t = \tan \frac{x}{2}, dx = \frac{2}{1+t^2} dt \right| = \int \frac{2-\frac{2t}{1+t^2}}{2+\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt = \int \frac{\frac{2+2t^2-2t}{1+t^2}}{\frac{2+2t^2+1-t^2}{1+t^2}} \frac{2}{1+t^2} dt = \\ &= \int \frac{2t^2-2t+2}{1+t^2} \cdot \frac{1+t^2}{t^2+3} \cdot \frac{2}{1+t^2} dt = \int \frac{2(2t^2-2t+2)}{(t^2+1)(t^2+3)} dt \end{aligned}$$

Parciální zlomky:

$$\begin{aligned} \frac{4t^2-4t+4}{(t^2+1)(t^2+3)} &= \frac{At+B}{t^2+1} + \frac{Ct+D}{t^2+3} \\ 4t^2-4t+4 &= (At+B)(t^2+3) + (Ct+D)(t^2+1) \\ 4t^2-4t+4 &= At^3+3At+Bt^2+3B+Ct^3+Ct+Dt^2+D \end{aligned}$$

Porovnáním koeficientů dostáváme:

$$\begin{aligned} 0 &= A+C \implies C = -A \xrightarrow{3} C = 2 \\ 4 &= B+D \xrightarrow{4} D = 4 \\ -4 &= 3A+C \xrightarrow{1} -4 = 2A \implies A = -2 \\ 4 &= 3B+D \xrightarrow{2} 0 = 2B \implies B = 0 \end{aligned}$$

$$\begin{aligned} \int \frac{4t^2-4t+4}{(t^2+1)(t^2+3)} dt &\stackrel{\text{lin.}}{=} \int \frac{-2t}{t^2+1} dt + \int \frac{2t+4}{t^2+3} dt = \\ &= \left| y = t^2+1, dy = 2t dt, z = t^2+3, dz = 2t dt \right| = \\ &= -\int \frac{1}{y} dy + \int \frac{1}{z} dz + 4 \int \frac{1}{t^2+3} dt = \left| u = \frac{t}{\sqrt{3}}, du = \frac{1}{\sqrt{3}} dt \right| = \\ &\stackrel{\text{⊛}}{=} -\log|y| + \log|z| + 4\sqrt{3} \int \frac{1}{u^2+1} du = \\ &\stackrel{\text{c}}{=} -\log|t^2+1| + \log|t^2+3| + 4\sqrt{3} \arctan u \stackrel{\text{c}}{=} \log \frac{t^2+3}{t^2+1} + 4\sqrt{3} \arctan \frac{t}{\sqrt{3}} = \\ &\stackrel{\text{c}}{=} \log \frac{\tan^2 \frac{x}{2} + 3}{\tan^2 \frac{x}{2} + 1} + 4\sqrt{3} \arctan \frac{\tan \frac{x}{2}}{\sqrt{3}} \\ &\quad (\text{⊛}): dx = \sqrt{3} \cdot \frac{1}{\sqrt{3}} dx = \sqrt{3} dt \end{aligned}$$

Příklad 1 (i) $\int \frac{\sin^3 x}{1+4\cos^2 x+3\sin^2 x} dx$
Vnitřek integrálu je definován všude.

Výběr substituce:

Chci: $R(\sin x, \cos x) = \frac{\sin^3 x}{1+4\cos^2 x+3\sin^2 x}$. Tj. $R(u, v) = \frac{u^3}{1+4v^2+3u^2}$.

Výraz	Čemu je roven	Čemu chci, aby se to rovnalo	Platí daný vztah?
$R(-\sin x, \cos x)$	$\frac{(-\sin x)^3}{1+4\cos^2 x+3(-\sin x)^2}$	$-\frac{\sin^3 x}{1+4\cos^2 x+3\sin^2 x}$	ano

Z Pythagorovy věty plyne první rovnost níže.

$$\begin{aligned}
 \int \frac{\sin^3 x}{1+4\cos^2 x+3\sin^2 x} dx &= \int \frac{\sin^3 x}{4+\cos^2 x} dx = |t = \cos x, dt = -\sin x dx| = \int \frac{-(1-t^2)}{4+t^2} dt = \\
 &= \int \frac{t^2-1}{t^2+4} dt = \int \frac{t^2-1+5-5}{t^2+4} dt \stackrel{\text{lin.}}{=} \int 1 dt - 5 \int \frac{1}{t^2+4} dt = \\
 &= t - \frac{5}{4} \int \frac{1}{\left(\frac{t}{2}\right)^2+1} dt = \left| y = \frac{t}{2}, dy = \frac{1}{2} dt \right| = t - \frac{5}{4} \cdot 2 \int \frac{1}{y^2+1} dy = \\
 &\stackrel{c}{=} t - \frac{5}{2} \arctan y \stackrel{c}{=} t - \frac{5}{2} \arctan \frac{t}{2} \stackrel{c}{=} \cos x - \frac{5}{2} \arctan \frac{\cos x}{2}
 \end{aligned}$$

Příklad 1 (j) $\int \frac{3\sin^2 x + \cos^2 x}{\sin^2 x + 3\cos^2 x} dx$

Vnitřek integrálu je zřejmě definován všude.

Výběr substituce:

Chci: $R(\sin x, \cos x) = \frac{3\sin^2 x + \cos^2 x}{\sin^2 x + 3\cos^2 x}$. Tj. $R(u, v) = \frac{3u^2+v^2}{u^2+3v^2}$.

Vidíme, že je splněn vztah $R(-\sin x, -\cos x) = R(\sin x, \cos x)$.

Proto volíme substituci $t = \tan x$. Integrujeme tedy jen na intervalech $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi), k \in \mathbb{Z}$.

Níže první rovnost plyne z Pythagorovy věty.

$$\begin{aligned}
 \int \frac{3\sin^2 x + \cos^2 x}{\sin^2 x + 3\cos^2 x} dx &= \int \frac{2\sin^2 x + 1}{1+2\cos^2 x} dx = \left| t = \tan x, dx = \frac{1}{t^2+1} dt \right| = \int \frac{2\frac{t^2}{1+t^2} + 1}{1+2\frac{1}{1+t^2}} \cdot \frac{1}{t^2+1} dt = \\
 &= \int \frac{2t^2+1+t^2}{1+t^2} \cdot \frac{1+t^2}{1+t^2+2} \cdot \frac{1}{t^2+1} dt = \int \frac{3t^2+1}{(t^2+3)(t^2+1)} dt
 \end{aligned}$$

Parciální zlomky:

$$\begin{aligned}
 \frac{3t^2+1}{(t^2+3)(t^2+1)} &= \frac{At+B}{t^2+3} + \frac{Ct+D}{t^2+1} \\
 3t^2+1 &= (At+B)(t^2+1) + (Ct+D)(t^2+3) \\
 3t^2+1 &= At^3+At+Bt^2+B+Ct^3+3Ct+Dt^2+3D
 \end{aligned}$$

Porovnáním koeficientů dostáváme:

$$\begin{aligned}
 0 &= A+C \\
 3 &= B+D \stackrel{4}{\implies} 3 = B-1 \implies B=4 \\
 0 &= A+3C \stackrel{1}{\implies} 0 = 2C \implies C=0 \stackrel{1}{\implies} A=0 \\
 1 &= B+3D \stackrel{2}{\implies} 1 = 3+2D \implies 2D = -2 \implies D = -1
 \end{aligned}$$

$$\begin{aligned}
& \int \frac{3t^2 + 1}{(t^2 + 3)(t^2 + 1)} dt \stackrel{\text{lin.}}{=} 4 \int \frac{1}{t^2 + 3} dt - \int \frac{1}{t^2 + 1} dt = \frac{4}{3} \int \frac{1}{\left(\frac{t}{\sqrt{3}}\right)^2 + 1} dt - \arctan t = \\
& = \left| u = \frac{t}{\sqrt{3}}, du = \frac{1}{\sqrt{3}} dt \right| = \frac{4\sqrt{3}}{3} \int \frac{1}{y^2 + 1} dy - \arctan t \stackrel{c}{=} \frac{4\sqrt{3}}{3} \arctan y - \arctan t = \\
& \stackrel{c}{=} \frac{4\sqrt{3}}{3} \arctan \frac{t}{\sqrt{3}} - \arctan t \stackrel{c}{=} \frac{4\sqrt{3}}{3} \arctan \frac{\tan x}{\sqrt{3}} - \arctan \tan x \stackrel{c}{=} \frac{4\sqrt{3}}{3} \arctan \frac{\tan x}{\sqrt{3}} - x
\end{aligned}$$

Příklad 3 (a) $\int \frac{\log x}{x - x \log x} dx$
Podmínky: $x > 0, x \neq 0, x \neq e$.

$$\begin{aligned}
& \int \frac{\log x}{x - x \log x} dx = \int \frac{1}{x} \frac{\log x}{1 - \log x} dx = \left| y = \log x, dy = \frac{1}{x} dx \right| = \int \frac{y}{1 - y} dy = \\
& = \int \frac{y - 1 + 1}{1 - y} dy \stackrel{\text{lin.}}{=} \int -1 dy + \int \frac{-1}{y - 1} dy = |u = y - 1, du = dy| = -y - \int \frac{1}{u} du = \\
& \stackrel{c}{=} -y - \log |u| \stackrel{c}{=} -y - \log |y - 1| \stackrel{c}{=} -\log x - \log |\log x - 1|
\end{aligned}$$

Příklad 3 (b) $\int \frac{1}{e^{2x} + e^x - 2} dx$
Podmínky: $e^{2x} + e^x - 2 = (e^x - 1)(e^x + 2) \neq 0 \implies x \neq 0$.

$$\int \frac{1}{e^{2x} + e^x - 2} dx = \int \frac{1}{(e^x)^2 + e^x - 2} dx = |y = e^x, dy = e^x dx| = \int \frac{1}{y(y^2 + y - 2)} dy$$

Parciální zlomky:

$$\begin{aligned}
& \frac{1}{y(y^2 + y - 2)} = \frac{1}{y(y + 2)(y - 1)} = \frac{A}{y} + \frac{B}{y + 2} + \frac{C}{y - 1} \\
& 1 = A(y^2 + y - 2) + By(y - 1) + Cy(y + 2) \\
& y = 0 \implies 1 = -2A \implies A = -\frac{1}{2} \\
& y = -2 \implies 1 = -2B(-2 - 1) \implies 1 = 6B \implies B = \frac{1}{6} \\
& y = 1 \implies 1 = 3C \implies C = \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
& \int \frac{1}{y(y^2 + y - 2)} dy \stackrel{\text{lin.}}{=} -\frac{1}{2} \int \frac{1}{y} dy + \frac{1}{6} \int \frac{1}{y + 2} dy + \frac{1}{3} \int \frac{1}{y - 1} dy = \\
& \stackrel{c}{=} -\frac{1}{6} \log |y| + \frac{1}{6} \log |y + 2| + \frac{1}{3} \log |y - 1| = \\
& \stackrel{c}{=} -\frac{1}{2} \log(e^x) + \frac{1}{6} \log(e^x + 2) + \frac{1}{3} \log |e^x - 1| \stackrel{c}{=} -\frac{1}{2} x + \frac{1}{6} \log(e^x + 2) + \frac{1}{3} \log |e^x - 1|
\end{aligned}$$

Poznámka: symbolem $\log$ máme na mysli přirozený logaritmus.

Příklad 3 (c) $\int \frac{\sqrt[4]{x}}{\sqrt[4]{x} + \sqrt{x}} dx$

Podmínky: $x > 0$.

$$\begin{aligned} \int \frac{\sqrt[4]{x}}{\sqrt[4]{x} + \sqrt{x}} dx &= \left| y = \sqrt[4]{x}, dy = \frac{1}{4} x^{-\frac{3}{4}} dx \right| = \int \frac{y}{y + y^2} \cdot 4y^3 dy \stackrel{\text{lin.}}{=} 4 \int \frac{y^3}{y + 1} dy = \\ &= 4 \int y^2 - y + 1 - \frac{1}{y + 1} dy \stackrel{c}{=} \frac{4}{3} y^3 - 2y^2 + 4y - \log |y + 1| \stackrel{c}{=} \frac{4}{3} \sqrt{x^3} - 2\sqrt{x} + 4\sqrt[4]{x} - \log |\sqrt[4]{x} + 1| \end{aligned}$$

Příklad 3 (d) $\int \frac{\sqrt{2x+3}+x}{\sqrt{2x+3}-x} dx$

Podmínky: $2x + 3 \geq 0 \implies x \geq -\frac{3}{2}$ a $2x + 3 - x^2 = 0 \implies x \neq 3, x \neq -1$. Vnitřek integrálu je tedy definován na $[-\frac{3}{2}, -1) \cup (-1, 3) \cup (3, \infty)$. Má tedy smysl integrál uvažovat na intervalech $(-\frac{3}{2}, -1), (-1, 3), (3, \infty)$.

$$\begin{aligned} \int \frac{\sqrt{2x+3}+x}{\sqrt{2x+3}-x} dx &= \left| y = \sqrt{2x+3}, \frac{y^2-3}{2} = x, dy = \frac{1}{\sqrt{2x+3}} dx \right| = \int \frac{y + \frac{y^2-3}{2}}{y - \frac{y^2-3}{2}} \cdot y dy = \\ &= \int \frac{2y + y^2 - 3}{2y - y^2 + 3} y dy = \int \frac{-y^3 - 2y^2 + 3y}{y^2 - 2y - 3} dy = \int -y - 4 + \frac{-8y - 12}{y^2 - 2y - 3} dy \end{aligned}$$

Parciální zlomky:

$$\begin{aligned} \frac{-8y - 12}{y^2 - 2y - 3} &= \frac{-8y - 12}{(y - 3)(y + 1)} = \frac{A}{y - 3} + \frac{B}{y + 1} \\ -8y - 12 &= A(y + 1) + B(y - 3) \\ y = -1 &\implies 8 - 12 = -4B \implies B = 1 \\ y = 3 &\implies -24 - 12 = 4A \implies A = -\frac{36}{4} = -9 \end{aligned}$$

$$\begin{aligned} \int -y - 4 + \frac{-8y - 12}{y^2 - 2y - 3} dy &\stackrel{\text{lin.}}{=} -\frac{y^2}{2} - 4y - 9 \int \frac{1}{y - 3} dy + \int \frac{1}{y + 1} dy = \\ &\stackrel{c}{=} -\frac{y^2}{2} - 4y - 9 \log |y - 3| + \log |y + 1| = \\ &\stackrel{c}{=} -\frac{2x + 3}{2} - 4\sqrt{2x + 3} - 9 \log |\sqrt{2x + 3} - 3| + \log (\sqrt{2x + 3} + 1) \end{aligned}$$

Příklad 3 (e) $\int \frac{e^{4x} + e^{2x}}{e^{3x} - 1} dx$

Podmínky: $e^{3x} \neq 1 \implies 3x \neq 0 \implies x \neq 0$

$$\begin{aligned} \int \frac{e^{4x} + e^{2x}}{e^{3x} - 1} dx &= |y = e^x, dy = e^x dx| = \int \frac{y^3 + y}{y^3 - 1} dy = \int \frac{y^3 - 1 + y + 1}{y^3 - 1} dy = \\ &\stackrel{\text{lin.}}{=} y + \int \frac{y + 1}{y^3 - 1} dy \end{aligned}$$

Parciální zlomky:

$$\begin{aligned}\frac{y+1}{y^3-1} &= \frac{y+1}{(y-1)(y^2+y+1)} = \frac{A}{y-1} + \frac{By+C}{y^2+y+1} \\ y+1 &= A(y^2+y+1) + (By+C)(y-1) \\ y=1 &\implies 2=3A \implies A=\frac{2}{3} \\ y+1 &= \frac{2}{3}y^2 + \frac{2}{3}y + \frac{2}{3} + By^2 - By + Cy - C \\ y+1 &= y^2\left(\frac{2}{3}+B\right) + y\left(\frac{2}{3}-B+C\right) + \frac{2}{3}-C\end{aligned}$$

Porovnáním koeficientů dostáváme:

$$\begin{aligned}0 &= \frac{2}{3} + B \implies B = -\frac{2}{3} \\ 1 &= \frac{2}{3} - B + C \\ 1 &= \frac{2}{3} - C \implies C = \frac{2}{3} - 1 = -\frac{1}{3}\end{aligned}$$

Máme tedy: $A = \frac{2}{3}, B = -\frac{2}{3}, C = -\frac{1}{3}$.

$$\begin{aligned}y + \int \frac{y+1}{y^3-1} dy &= y + \frac{2}{3} \int \frac{1}{y-1} dy - \frac{1}{3} \int \frac{2y+1}{y^2+y+1} = |u = y^2 + y + 1, du = (2y+1) dy| = \\ &= y + \frac{2}{3} \log|y-1| - \frac{1}{3} \int \frac{1}{u} du \stackrel{c}{=} y + \frac{2}{3} \log|y-1| - \frac{1}{3} \log|y^2+y+1| = \\ &\stackrel{c}{=} e^x + \frac{2}{3} \log|e^x-1| - \frac{1}{3} \log(e^{2x}+e^x+1)\end{aligned}$$

Příklad 3 (f) $\int \frac{1}{x \log x \cdot \log(\log x)} dx$

$$\begin{aligned}\int \frac{1}{x \log x \cdot \log(\log x)} dx &= \left| y = \log x, dy = \frac{1}{x} dx \right| = \int \frac{1}{y \log(y)} dy = \left| u = \log y, du = \frac{1}{y} dy \right| = \\ &= \int \frac{1}{u} du \stackrel{c}{=} \log|u| \stackrel{c}{=} \log|\log y| \stackrel{c}{=} \log|\log(\log x)|\end{aligned}$$

Příklad 3 (g) $\int \frac{1}{x} \sqrt{\frac{x+2}{x-3}} dx$

Výběr substituce: $R\left(x, \sqrt{\frac{x+2}{x-3}}\right) = \frac{1}{x} \sqrt{\frac{x+2}{x-3}}, (R(u, v) = \frac{v}{u})$

Provedeme substituci $t = \sqrt{\frac{x+2}{x-3}}$. Vyjádříme nejdříve x v závislosti na t .

$$t^2 = \frac{x+2}{x-3} \implies t^2(x-3) = x+2 \implies x(t^2-1) = 2+3t^2 \implies x = \frac{3t^2+2}{t^2-1}$$

Zderivováním získaného vztahu dostaneme vztah pro dx a dt :

$$dx = \frac{6t(t^2-1) - (3t^2+2)(2t)}{(t^2-1)^2} dt = \frac{6t^3 - 6t - 6t^3 - 4t}{(t^2-1)^2} dt = \frac{-10t}{(t^2-1)^2} dt$$

$$\begin{aligned} \int \frac{1}{x} \sqrt{\frac{x+2}{x-3}} dx &= \left| t = \sqrt{\frac{x+2}{x-3}}, x = \frac{3t^2+2}{t^2-1}, dx = \frac{-10t}{(t^2-1)^2} dt \right| = \int \frac{t^2-1}{3t^2+2} \cdot t \cdot \frac{-10t}{(t^2-1)^2} dt = \\ &= \int \frac{-10t^2}{(t^2-1)(3t^2+2)} dt \end{aligned}$$

Parciální zlomky:

$$\begin{aligned} \frac{-10t^2}{(t^2-1)(3t^2+2)} &= \frac{-10t^2}{(t-1)(t+1)(3t^2+2)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{Ct+D}{3t^2+2} \\ -10t^2 &= (At+A)(3t^2+2) + (Bt-B)(3t^2+2) + (Ct+D)(t^2-1) \\ t=1 &\implies -10 = 2A(3+2) \implies A = -1 \\ t=-1 &\implies -10 = -10B \implies B = 1 \\ -10t^2 &= -3t^3 - 2t - 3t^2 - 2 + 3t^3 + 2t - 3t^2 - 2 + Ct^3 - Ct + Dt^2 - D \\ -10t^2 &= Ct^3 + (D-6)t^2 - Ct - 4 - D \end{aligned}$$

Porovnáním koeficientů dostáváme:

$$\begin{aligned} C &= 0 \\ -10 &= D - 6 \implies D = -4 \\ 0 &= -C \\ 0 &= -4 - D \end{aligned}$$

Řešení soustavy: $A = -1, B = 1, C = 0, D = -4$.

$$\begin{aligned} \int \frac{-10t^2}{(t^2-1)(3t^2+2)} dt &\stackrel{\text{lin.}}{=} - \int \frac{1}{t-1} dt + \int \frac{1}{t+1} dt - 4 \int \frac{1}{3t^2+2} dt = \\ &\stackrel{c}{=} -\log|t-1| + \log|t+1| - \frac{2\sqrt{2}}{\sqrt{3}} \arctan \sqrt{\frac{3}{2}}t = \\ &\stackrel{c}{=} -\log \left| \sqrt{\frac{x+2}{x-3}} - 1 \right| + \log \left| \sqrt{\frac{x+2}{x-3}} + 1 \right| - \frac{2\sqrt{2}}{\sqrt{3}} \arctan \left(\sqrt{\frac{3}{2}} \sqrt{\frac{x+2}{x-3}} \right) \end{aligned}$$

Poslední integrály jsme spočetli jako již mnohokrát dříve - substitucí a úpravou výrazu...

Příklad 3 (h) $\int \frac{1-\sqrt{x+1}}{1-\sqrt[3]{x+1}} dx$

Výběr substituce:

Převědeme nejdříve všechny odmocniny na mocniny též odmocniny: $\sqrt{x} = (\sqrt[6]{x})^3, \sqrt[3]{x} = (\sqrt[6]{x})^2$

$$R(x, \sqrt[6]{x}) = \frac{1-\sqrt{x+1}}{1-\sqrt[3]{x+1}}, (R(u, v) = \frac{1-v^3}{1-v^2})$$

$$\begin{aligned}
\int \frac{1 - \sqrt{x+1}}{1 - \sqrt[3]{x+1}} dx &= \left| t = \sqrt[6]{x+1}, t^3 = \sqrt{x+1}, t^2 = \sqrt[3]{x+1}, x = t^6 - 1 \implies dx = 6t^5 dt \right| = \\
&= \int \frac{1 - t^3}{1 - t^2} 6t^5 dt = \int \frac{6t^5(t-1)(t^2+t+1)}{(t+1)(t-1)} dt = \int \frac{6t^7 + 6t^6 + 6t^5}{t+1} dt = \\
&= \int 6t^6 + 6t^4 - 6t^3 + 6t^2 - 6t + 6 - \frac{6}{t+1} dt = \\
&\stackrel{c}{=} \frac{6}{7}t^7 + \frac{6}{5}t^5 - \frac{6}{4}t^4 + \frac{6}{3}t^3 - \frac{6}{2}t^2 + 6t - 6 \log|t+1| = \\
&\stackrel{c}{=} \frac{6}{7}\sqrt[6]{(x+1)^7} + \frac{6}{5}\sqrt[6]{(x+1)^5} - \frac{3}{2}\sqrt[6]{(x+1)^4} + 2\sqrt[6]{(x+1)^3} - 3\sqrt[6]{(x+1)^2} + 6\sqrt[6]{x+1} - \\
&\quad - 6 \log(\sqrt[6]{x+1} + 1)
\end{aligned}$$

Příklad 3 (i) $\int \frac{1}{x(\log^3 x - 1)} dx$

$$\int \frac{1}{x(\log^3 x - 1)} dx = \left| y = \log x, dy = \frac{1}{x} dx \right| = \int \frac{1}{y^3 - 1} dy$$

Parciální zlomky:

$$\begin{aligned}
\frac{1}{y^3 - 1} &= \frac{1}{(y-1)(y^2 + y + 1)} = \frac{A}{y-1} + \frac{By + C}{y^2 + y + 1} \\
1 &= Ay^2 + Ay + A + By^2 - By + Cy - C
\end{aligned}$$

Porovnáním koeficientů dostáváme:

$$\begin{aligned}
0 &= A + B \\
0 &= A - B + C \\
1 &= A - C
\end{aligned}$$

Řešení soustavy: $A = \frac{1}{3}, B = -\frac{1}{3}, C = -\frac{2}{3}$.

$$\begin{aligned}
\int \frac{1}{y^3 - 1} dy &\stackrel{\text{lin.}}{=} \frac{1}{3} \int \frac{1}{y-1} dy - \frac{1}{3} \int \frac{y+2}{y^2 + y + 1} dy = \left| u = y^2 + y + 1, du = (2y+1) dy \right| = \\
&= \frac{1}{3} \log|y-1| - \frac{1}{6} \int \frac{2y+4-3+3}{y^2 + y + 1} dy = \frac{1}{3} \log|y-1| - \frac{1}{6} \int \frac{1}{u} du - \frac{1}{2} \int \frac{1}{y^2 + y + 1} dy = \\
&= \frac{1}{3} \log|y-1| - \frac{1}{6} \log|u| - \frac{1}{2} \int \frac{1}{(y+\frac{1}{2})^2 + \frac{3}{4}} dy = \frac{1}{3} \log|y-1| - \frac{1}{6} \log|u| - \frac{2}{3} \int \frac{1}{\left(\frac{y+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2 + 1} dy = \\
&\stackrel{c}{=} \frac{1}{3} \log|y-1| - \frac{1}{6} \log|u| - \frac{2}{3} \sqrt{\frac{3}{4}} \arctan \left(\frac{y+\frac{1}{2}}{\sqrt{\frac{3}{4}}} \right) = \\
&\stackrel{c}{=} \frac{1}{3} \log|y-1| - \frac{1}{6} \log|y^2 + y + 1| - \frac{1}{\sqrt{3}} \arctan \left(\frac{1}{\sqrt{3}} \left(y + \frac{1}{2} \right) \right) = \\
&\stackrel{c}{=} \frac{1}{3} \log|\log x - 1| - \frac{1}{6} \log|\log^2 x + \log x + 1| - \frac{1}{\sqrt{3}} \arctan \left(\frac{2}{\sqrt{3}} \left(\log x + \frac{1}{2} \right) \right)
\end{aligned}$$

Příklad 3 (j) $\int \sqrt{\frac{x-1}{x+2}} dx$

Výběr substituce: $R\left(x, \sqrt{\frac{x-1}{x+2}}\right) =, (R(u, v) = v)$

Provedeme substituci: $t = \sqrt{\frac{x-1}{x+2}}$. Odvoďme x , dx .

$$t^2 = \frac{x-1}{x+2} \implies t^2(x+2) = x-1 \implies x(t^2-1) = -1-2t^2 \implies x = \frac{-2t^2-1}{t^2-1}$$
$$x = \frac{-2t^2-1}{t^2-1} \implies dx = \frac{-4t(t^2-1) - (-2t^2-1)2t}{(t^2-1)^2} dt = \frac{-4t^3+4t+4t^3+2t}{(t^2-1)^2} dt = \frac{6t}{(t^2-1)^2} dt$$

$$\int \sqrt{\frac{x-1}{x+2}} dx = \left| t = \sqrt{\frac{x-1}{x+2}}, dx = \frac{6t}{(t^2-1)^2} dt \right| = \int \frac{6t^2}{(t^2-1)^2} dt$$

Parciální zlomky:

$$\frac{6t^2}{(t^2-1)^2} = \frac{6t^2}{(t-1)^2(t+1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$$
$$6t^2 = (At - A + B)(t^2 + 2t + 1) + (Ct + C + D)(t^2 - 2t + 1)$$
$$t = 1 \implies 6 = 4B \implies B = \frac{3}{2}$$
$$t = -1 \implies 6 = 4D \implies D = \frac{3}{2}$$
$$6t^2 = At^3 + 2At^2 + At - At^2 - 2At - A + \frac{3}{2}t^2 + 3t + \frac{3}{2} + Ct^3 - 2Ct^2 + Ct + Ct^2 - 2Ct +$$
$$+ C + \frac{3}{2}t^2 - 3t + \frac{3}{2}$$

Porovnáním koeficientů dostáváme:

$$0 = A + C \implies A = -C \xrightarrow{2} A = \frac{3}{2}$$
$$6 = A + 3 - C \xrightarrow{1} 3 = -2C \implies C = -\frac{3}{2}$$
$$0 = -A - C$$
$$0 = -A + 3 + C$$

Řešením soustavy je: $A = \frac{3}{2}, B = \frac{3}{2}, C = -\frac{3}{2}, D = \frac{3}{2}$.

$$\int \frac{6t^2}{(t^2-1)^2} dt \stackrel{\text{lin.}}{=} \frac{3}{2} \int \frac{1}{t-1} dt + \frac{3}{2} \int \frac{1}{(t-1)^2} dt - \frac{3}{2} \int \frac{1}{t+1} dt + \frac{3}{2} \int \frac{1}{(t+1)^2} dt =$$
$$\stackrel{c}{=} \frac{3}{2} \log|t-1| - \frac{3}{2} \frac{1}{t+1} - \frac{3}{2} \log|t+1| - \frac{3}{2} \frac{1}{t+1} =$$
$$\stackrel{c}{=} \frac{3}{2} \log \left| \sqrt{\frac{x-1}{x+2}} - 1 \right| - \frac{3}{2} \frac{1}{\sqrt{\frac{x-1}{x+2}} + 1} - \frac{3}{2} \log \left| \sqrt{\frac{x-1}{x+2}} + 1 \right| - \frac{3}{2} \frac{1}{\sqrt{\frac{x-1}{x+2}} + 1}$$

Příklad 3 (k) $\int \frac{1}{x} \sqrt{x^2 - 2x} \, dx$

Výběr substituce: $R(x, \sqrt{x^2 - 2x}) = \frac{1}{x} \sqrt{x^2 - 2x}, (R(u, v) = \frac{v}{u})$

Platí, že $x^2 - 2x = x(x - 2)$. Provedeme substituci $t = \sqrt{\frac{x}{x-2}}$ (dle nápovědy z teorie k 10. cvičení).

$$t^2 = \frac{x}{x-2} \implies t^2 x - 2t^2 = x \implies x = \frac{2t^2}{t^2 - 1} \implies dx = \frac{-4t}{(t^2 - 1)^2}$$

$$\begin{aligned} \int \frac{1}{x} \sqrt{x^2 - 2x} \, dx &= \left| t = \sqrt{\frac{x}{x-2}}, dx = \frac{-4t}{(t^2 - 1)^2} \right| = \int \frac{t^2 - 1}{2t^2} \sqrt{\left(\frac{2t^2}{t^2 - 1}\right)^2 - 2 \frac{2t^2}{t^2 - 1}} \frac{-4t}{(t^2 - 1)^2} dt = \\ &= \int \frac{-2}{t(t^2 - 1)} \sqrt{\frac{4t^4 - 4t^2(t^2 - 1)}{(t^2 - 1)^2}} dt = \int \frac{4t}{t(t^2 - 1)^2} dt = \int -\frac{1}{t+1} - \frac{1}{(t+1)^2} + \frac{1}{t-1} - \frac{1}{(t-1)^2} dt = \\ &\stackrel{c}{=} -\log|t+1| + \frac{1}{t+1} + \log|t-1| + \frac{1}{t-1} = \\ &\stackrel{c}{=} -\log\left|\sqrt{\frac{x}{x-2}} + 1\right| + \frac{1}{\sqrt{\frac{x}{x-2}} + 1} + \log\left|\sqrt{\frac{x}{x-2}} - 1\right| + \frac{1}{\sqrt{\frac{x}{x-2}} - 1} \end{aligned}$$

Příklad 3 (l) $\int \frac{1}{x(1+2\sqrt{x}+\sqrt[3]{x})} \, dx$

Výběr substituce:

Opět převedeme na mocniny stejné odmocniny.

$$R(x, \sqrt[6]{x}) = \frac{1}{x(1+2\sqrt{x}+\sqrt[3]{x})}, (R(u, v) = \frac{1}{u(1+2v^3+v^2)})$$

$$\begin{aligned} \int \frac{1}{x(1+2\sqrt{x}+\sqrt[3]{x})} \, dx &= |t = \sqrt[6]{x}, t^3 = \sqrt{x}, t^2 = \sqrt[3]{x}, x = t^6 \implies dx = 6t^5 \, dt| = \\ &= \int \frac{1}{t^6(1+2t^3+t^2)} 6t^5 \, dt = \int \frac{6}{t(2t^3+t^2+1)} \, dt \end{aligned}$$

Polynom $2t^3 + t^2 + 1$ má zřejmě kořen -1 . Z toho plyne následující.

$$\begin{aligned} \frac{6}{t(2t^3+t^2+1)} &= \frac{6}{t(t+1)(2t^2-t+1)} = \frac{A}{t} + \frac{B}{t+1} + \frac{Ct+D}{2t^2-t+1} \\ 6 &= (At+A)(2t^2-t+1) + Bt(2t^2-t+1) + (Ct+D)(t^2+t) \\ t=0 &\implies 6=A \\ t=-1 &\implies 6=-4B \implies B=-\frac{3}{2} \\ 6 &= 12t^3 - 6t^2 + 6t + 12t^2 - 6t + 6 - 3t^3 - \frac{3}{2}t^2 + \frac{3}{2}t + Ct^3 + Ct^2 + Dt^2 + Dt \end{aligned}$$

Porovnáním koeficientů dostáváme:

$$0 = 9 + C \implies C = -9$$

$$0 = \frac{15}{2} + C + D \xrightarrow{1} 9 - \frac{15}{2} = D \implies D = \frac{3}{2}$$

$$0 = -\frac{3}{2} + D$$

$$6 = 6$$

Řešení: $A = 6, B = -\frac{3}{2}, C = -9, D = \frac{3}{2}$

$$\begin{aligned} \int \frac{6}{t(2t^3 + t^2 + 1)} dt &\stackrel{\text{lin.}}{=} 6 \int \frac{1}{t} dt - \frac{3}{2} \int \frac{1}{t+1} dt + \int \frac{-9t + \frac{3}{2}}{2t^2 - t + 1} dt = \\ &= \left| y = 2t^2 - t + 1, \quad dy = (4t - 1) dt \right| = \\ &= 6 \log |t| - \frac{3}{2} \log |t+1| - \frac{9}{4} \int \frac{4t - \frac{2}{3} - \frac{1}{3} + \frac{1}{3}}{2t^2 - t + 1} dt = \\ &= 6 \log |t| - \frac{3}{2} \log |t+1| - \frac{9}{4} \int \frac{1}{y} dy - \frac{9}{4} \cdot \frac{1}{3} \int \frac{1}{2t^2 - t + 1} dt = \\ &= 6 \log |t| - \frac{3}{2} \log |t+1| - \frac{9}{4} \log |y| - \frac{3}{8} \int \frac{1}{t^2 - \frac{1}{2}t + \frac{1}{2}} dt = \\ &= 6 \log |t| - \frac{3}{2} \log |t+1| - \frac{9}{4} \log |y| - \frac{3}{8} \int \frac{1}{\left(t - \frac{1}{4}\right)^2 + \frac{7}{16}} dt = \\ &= 6 \log |t| - \frac{3}{2} \log |t+1| - \frac{9}{4} \log |y| - \frac{3}{8} \cdot \frac{16}{7} \int \frac{1}{\left(\frac{t - \frac{1}{4}}{\sqrt{\frac{7}{16}}}\right)^2 + 1} dt = \\ &= \left| z = \frac{4}{\sqrt{7}} \left(t - \frac{1}{4}\right), \quad dz = \frac{4}{\sqrt{7}} dt \right| = \\ &= 6 \log |t| - \frac{3}{2} \log |t+1| - \frac{9}{4} \log |y| - \frac{6}{7} \cdot \frac{\sqrt{7}}{4} \int \frac{1}{z^2 + 1} dz = \\ &\stackrel{c}{=} 6 \log |t| - \frac{3}{2} \log |t+1| - \frac{9}{4} \log |y| - \frac{3}{2\sqrt{7}} \arctan z = \\ &\stackrel{c}{=} 6 \log |t| - \frac{3}{2} \log |t+1| - \frac{9}{4} \log |2t^2 - t + 1| - \frac{3}{2\sqrt{7}} \arctan \frac{4t - 1}{\sqrt{7}} = \\ &\stackrel{c}{=} 6 \log \sqrt[6]{x} - \frac{3}{2} \log (\sqrt[6]{x} + 1) - \frac{9}{4} \log |2\sqrt[3]{x} - \sqrt[6]{x} + 1| - \frac{3}{2\sqrt{7}} \arctan \frac{4\sqrt[6]{x} - 1}{\sqrt{7}} \end{aligned}$$